

Differential Equation of the first order and first degree

Methods of solving: —

A diff. eqn. can be written in the form

$$Mdx + Ndy = 0$$

where M and N are fns. of x and y or arbitrary constants.

We shall adopt different methods in solving the diff. eqn. of the first order and first degree.

- (i) By the method of variables separable.
 - (ii) (a) Homogeneous equations in x and y .
(b) Non-homogeneous equations in x and y .
 - (iii) (a) Linear equations.
(b) Equations reducible to linear form.
 - (iv) (a) Exact differential equations
(b) Equations reducible to exact equations.
- [We shall take these methods up one by one.]

• Variable separable: —

If the equation $Mdx + Ndy = 0$ where M and N are fns. of x and y or arbitrary constant can be put in the form

$$f(x)dx + \phi(y)dy = 0$$

where $f(x)$ is a fn. of x only, and $\phi(y)$ is a fn. of y only, then we say that variables are separable. In such case by using direct integration we can find the solution of diff. eqn.

Q. (1). Solve the differential equation

$$\frac{dy}{dx} = \frac{x(1+y^2)}{y(1+x^2)}$$

Solution:— Here, we see that variables are separable. The given equation can be written in the form

$$\frac{x}{(1+x^2)} dx = \frac{y}{(1+y^2)} dy$$

Now, integrating of both the sides,

$$\int \frac{x}{1+x^2} dx = \int \frac{y}{1+y^2} dy + C \quad [\text{where } C \text{ is some constant}]$$

$$\Rightarrow \frac{1}{2} \int \frac{2x}{1+x^2} dx = \frac{1}{2} \int \frac{2y}{1+y^2} + C$$

$$\Rightarrow \frac{1}{2} \log(1+x^2) = \frac{1}{2} \log(1+y^2) + C$$

$$\Rightarrow \frac{1}{2} [\log(1+x^2) - \log(1+y^2)] = C$$

$$\Rightarrow \log\left(\frac{1+x^2}{1+y^2}\right) = 2C = \log K \quad (\text{say})$$

$$\therefore \boxed{1+x^2 = K(1+y^2)} \rightarrow \text{Answer.}$$

Q. (2). Solve $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$

Solution:— The given equation be

$$\frac{dy}{dx} = e^{x-y} + x^2 e^{-y} = \frac{e^x}{e^y} + \frac{x^2}{e^y} = \frac{e^x + x^2}{e^y}$$

$\Rightarrow e^y dy = (e^x + x^2) dx$, Integrating, we get

$$\text{i.e. } \int e^y dy = \int e^x dx + \int x^2 dx + C$$

$$\Rightarrow \boxed{e^y = e^x + \frac{x^3}{3} + C} \rightarrow \text{Answer.}$$

Q. (3). Solve: $y + \frac{dy}{dx} (1+x^2) \tan^{-1} x = 0$

Solution:- The given equation can be written as

$$\frac{dy}{dx} = - \frac{y}{(1+x^2) \tan^{-1} x}$$

$$\Rightarrow -\frac{1}{y} dy = \frac{1}{(1+x^2) \tan^{-1} x} dx, \quad \text{Integrating}$$

$$\Rightarrow -\int \frac{1}{y} dy = \int \frac{1}{(1+x^2) \cdot \tan^{-1} x} \cdot dx + C \text{ (Constant)} \quad \text{--- (A)}$$

I_1 I_2

Now, $I_1 = -\int \frac{1}{y} dy = -e^y \dots \dots \dots (1)$

and $I_2 = \int \frac{1}{(1+x^2) \tan^{-1} x} dx \dots \dots \dots (2)$

Now, we put $\tan^{-1} x = z$ so that

$$\frac{1}{1+x^2} dx = dz$$

$$(2) \Rightarrow \int \frac{1}{z} dz = e^z = e^{\tan^{-1} x} \dots \dots \dots (3)$$

Now, (A) becomes [using (1) and (3)]

$$-e^y = e^{\tan^{-1} x} + C \Rightarrow C = e^{\tan^{-1} x} + e^y = e^{y \tan^{-1} x}$$

$$\therefore C = e^{y \tan^{-1} x}$$

Now, we can take e^k in stead of C , since e^k is also a some constant (number)

$$\therefore \boxed{K = y \tan^{-1} x} \rightarrow \text{Answer}$$

Q.(4). solve $\log\left(\frac{dy}{dx}\right) = ax + by$.

Solution:- Given equation be $\log\left(\frac{dy}{dx}\right) = ax + by$

$$\Rightarrow \frac{dy}{dx} = e^{ax+by} = e^{ax} \cdot e^{by}$$

$\Rightarrow e^{ax} dx = e^{-by} dy$, Integrating, we get

$$\frac{e^{ax}}{a} = -\frac{e^{-by}}{b} + C \Rightarrow \frac{e^{ax}}{a} + \frac{e^{-by}}{b} = C \text{ (Constant)}$$

$$\therefore \boxed{be^{ax} + ae^{-by} = K} \text{ (some constant)}$$

which is the required solution.

Q.(5). Solve: $\sec^2 x \tan x dx + \sec^2 y \tan y dy = 0$

Solution:- Given equation can be written as

$$\frac{\sec^2 x}{\tan x} dx = -\frac{\sec^2 y}{\tan y} dy, \text{ Integrating, we get}$$

$$\int \frac{\sec^2 x}{\tan x} dx = -\int \frac{\sec^2 y}{\tan y} dy + C, \text{ where } C \text{ is arbitrary constant.}$$

$$I_1 = \int \frac{\sec^2 x}{\tan x} dx, \text{ Putting } \tan x = z \text{ so that } \sec^2 x dx = dz$$

$$\text{Similarly, } \int \frac{\sec^2 y}{\tan y} dy, \text{ Putting } \tan y = t, \sec^2 y dy = dt$$

\therefore The above result becomes

$$C = \int \frac{\sec^2 x}{\tan x} dx + \int \frac{\sec^2 y}{\tan y} dy = \int \frac{1}{z} dz + \int \frac{1}{t} dt$$

$$\Rightarrow \log z + \log t = \log k \quad [\because \text{where } C = \log k \text{ (constant)}]$$

$$k = z \cdot t \quad \therefore \boxed{k = \tan x \cdot \tan y}$$

which is required solution.

Q. (6). Find the equation of the curve passing through the point $(0, 2)$ and having the gradient e^x at $P(x, y)$.

Solution: - We know that the

$$\text{gradient} = \frac{dy}{dx}$$

Hence, according to given question, we have

$$\frac{dy}{dx} = e^x \Rightarrow e^x dx = dy \Rightarrow \int e^x dx = \int dy + C$$

$$\Rightarrow e^x = y + C \quad \text{--- (1)}$$

Since the curve (1) passes through the point $(0, 2)$

Hence $x=0$ and $y=2$, then (1) becomes

$$e^0 = 2 + C \Rightarrow C = 1 - 2 = -1$$

Putting the value of C in (1), we get

$$e^x = -1 + C \quad \therefore \boxed{C = e^x + 1}$$

which is the required equation.

Q. (7). Find the equation of the curve which passes through the point $(2, 1)$ and whose slope at any point $P(x, y)$ is x/y .

Solution: - Given that slope at any point (x, y)

$$\text{is } \frac{x}{y}, \text{ i.e. } \frac{dy}{dx} = \frac{x}{y} \Rightarrow y dy = x dx$$

$$\text{Integrating, } \frac{x^2}{2} = \frac{y^2}{2} + C \Rightarrow C = \frac{x^2 - y^2}{2} \Rightarrow x^2 - y^2 = 2C = K \text{ (say)}$$

$$\therefore K = x^2 - y^2 \quad \text{--- (1)}$$

$$\therefore (1) \text{ passes } (2, 1)$$

$$\therefore K = 4 - 1 = 3$$

\therefore The required eqn. is

$$\boxed{x^2 - y^2 = 3}$$

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01.06.2020.